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# The expected area of the Wiener sausage swept by a disc

Kôhei UCHIYAMA

Department of Mathematics, Tokyo Institute of Technology
Oh-okayama, Meguro Tokyo 152-8551
e-mail: uchiyama@math.titech.ac.jp

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**Abstract**

The expected areas of the Wiener sausages swept by a disc attached to the two-dimensional Brownian Bridge joining the origin to a point $x$ over a time interval $[0,t]$ are computed. It is proved that the leading term of the expectation is given by Ramanujan's function if $|\mathbf{x}| = O(\sqrt{t})$. The second term is also given explicitly when $|\mathbf{x}| = o(\sqrt{t})$. The corresponding result for unconditioned process is also obtained.

## 1 Introduction

Let $B_t$ be the standard two-dimensional Brownian motion started at the origin defined on a probability space $(\Omega, \mathcal{F}, P)$. Fixing $r > 0$ let $S_t^{(r)}$ be the Wiener sausage of radius $r$ and length $t$, namely it is the region swept by the disc of radius $r$ attached to $B_s$ at its center as $s$ runs from 0 to $t$:

$$S_t^{(r)} = \{\mathbf{z} \in \mathbf{R}^2 : |B_s - \mathbf{z}| < r \ \text{ for some } \ s \in [0,t]\}.$$

In this paper we compute the expectation of the area of $S_t^{(r)}$, which we denote by $\mathrm{Area}(S_t^{(r)})$, for Brownian motion conditioned to be at a prescribed point $\mathbf{x} \in \mathbf{R}^2$ at time $t$ as well as for free (unconditioned) Brownian motion. Define $N(\lambda)$, called Ramanujan's function ([2], [29]) or integral ([7], page 219), by

$$N(\lambda) = \int_0^\infty \frac{e^{-\lambda u}}{(\lg u)^2 + \pi^2} \cdot \frac{du}{u} \qquad (\lambda \geq 0).$$

Put $\kappa = 2e^{-2\gamma}$ where $\gamma = -\int_0^\infty e^{-u} \lg u \, du$ (Euler's constant). Let $E$ designate the expectation with respect to $P$. In this paper we prove the following two theorems.

**Theorem 1.1**

$$\begin{aligned}\frac{d}{dt}E[\mathrm{Area}(S_t^{(r)})] &= 2\pi N\Big(\frac{\kappa t}{r^2}\Big) + \frac{4\pi r^2}{t(\lg(t/r^2))^3}(1+o(1)) \qquad as \qquad t\to\infty,\\ &= r\sqrt{\frac{2\pi}{t}} + \frac{\pi}{2} + O\big(\sqrt{t}\big) \qquad as \qquad t \to 0.\end{aligned}$$

For $\mathbf{x}\in\mathbf{R}^2$ we write $\mathbf{x}^2$ for the square of Euclidian length $|\mathbf{x}|$.

**Theorem 1.2** *For each $M>1$, uniformly for $|\mathbf{x}|<M\sqrt{t}$, as $t\to\infty$*

$$E[\mathrm{Area}(S_t^{(r)})\,|\,B_t=\mathbf{x}] = 2\pi t N\Big(\frac{\kappa t}{r^2}\Big) + \frac{\pi\mathbf{x}^2}{(\lg t)^2}\Big[\lg\Big(\frac{t}{\mathbf{x}^2\vee 1}\Big)+O(1)\Big]+O(1),$$

*where the left-hand side is the conditional expectation conditioned on $B_t=\mathbf{x}$.*

The proof of Theorem 1.1 is performed by Laplace inversion with rather simple computations. The leading term in the formula of Theorem 1.2 is derived in a similar way to that in Theorem 1.1, while the identification of the second term of it is made by a delicate analysis unless $|\mathbf{x}|$ remains in a bounded set.

Remark 1. The function $N(t)$ admits the following asymptotic expansion in powers of $1/\lg t$ as $t\to\infty$:

$$N(t)\sim\frac{1}{\lg t}+\frac{-\gamma}{(\lg t)^2}+\frac{\gamma^2-\zeta(2)}{(\lg t)^3}+\frac{3\gamma\zeta(2)+(-\gamma)^3-2\zeta(3)}{(\lg t)^4}+\cdots, \tag{1.1}$$

where $\zeta(z)=\sum_{n=1}^\infty n^{-z}$. While it requires a tedious computation to derive the corresponding expansion of $N(\kappa t/r^2)$ directly from this one, actually there is a simple way to transform the expansion. For each $\alpha>0$ the expansion for $N(\alpha t)$ is obtained by simply replacing $-\gamma$ by $-\gamma-\lg\alpha$ in (1.1), so that the expansion up to the third order term becomes

$$N(\alpha t)\sim\frac{1}{\lg t}+\frac{-\gamma-\lg\alpha}{(\lg t)^2}+\frac{(\gamma+\lg\alpha)^2-\zeta(2)}{(\lg t)^3}+\cdots \tag{1.2}$$

Similarly the asymptotic expansion of $\frac{1}{t}\int_0^t N(\alpha s)ds$ is obtained by replacing $-\gamma$ by $1-\gamma-\lg\alpha$ and $-\zeta(k)$ by $1-\zeta(k)$ in (1.1), so that

$$\frac{1}{t}\int_0^t N(\alpha s)ds\sim\frac{1}{\lg t}+\frac{1-\gamma-\lg\alpha}{(\lg t)^2}+\frac{(1-\gamma-\lg\alpha)^2+1-\zeta(2)}{(\lg t)^3}+\cdots \tag{1.3}$$

(See Section 5 for these and related matters.) It also is noted that combining Theorems 1.1 and 1.2 yields

$$E[\mathrm{Area}(S_t^{(r)})]-E[\mathrm{Area}(S_t^{(r)})\,|\,B_t=0]=2\pi\int_0^t\Big[-\alpha sN'(\alpha s)\Big]ds+O(1)$$

as $t\to\infty$, where $\alpha=\kappa/r^2$. The asymptotic expansion of this difference is obtained from that of $E[\mathrm{Area}(S_t^{(r)})]/\lg t$ by a very simple rule (see Lemma 5.2 in Section 5).

Remark 2. It may be reasonable to compare the expected increment of the sausage at time $t$ with $2\pi rE[|B_t|]$ as $t\downarrow 0$. The former one is asymptotic to $2r\sqrt{2\pi t}$ according to

the second half of Theorem 1.1, while the latter equals $2\pi r\sqrt{\pi t/2}$, so that the ratio of the latter to the former equals $\pi/2$.

Asymptotic behavior of the area $\mathrm{Area}(S_t^{(r)})$ (or volume in higher dimensions) as $t \to \infty$ has long been studied from various points of view. It is a typical functional of Brownian paths that is non-Markovian and the standard limit theorems for it (the law of large numbers, the central limit theorems or the large deviations) have been of continued interest ([30], [16], [10], etc.) . The expectation of it is the total heat emitted in the time interval $[0, t]$ from the disc which is kept at the unit temperature. The sausage for conditioned process (with $\mathbf{x} = 0$) naturally arises in the study of the asymptotic estimate of the trace of the heat kernel on the plane with randomly scattered cooling discs kept at zero temperature and has been effectively used (cf. [13], [5]).

For free Brownian motion, some asymptotic expansions in negative powers of $\lg t$ of the expectation divided by $t$ for the sausage swept by an arbitrary (non-polar) compact subset of $\mathbf{R}^2$ are obtained by Spitzer [22] up to magnitude $o(1/(\lg t)^2)$ and by Le Gall [15] to any order (see also [17]). M. van den Berg and E. Bolthausen [3] compute the expectation for Brownian motion conditioned on returning to the origin at time $t$ for the disc case up to the error term of magnitude $O(t\sqrt{\lg\lg t})/(\lg t)^4$ and conjecture that their formula for the conditional expansion would be valid for the general compact set, $K$ say, of positive capacity if the radius of the disc is replaced by the logarithmic capacity of $K$ (in the usual normalization [1]), the situation already observed for the free Brownian case.

The conjecture stated in [3] has been verified in a very recent paper [19] by I. McGillivray: in fact he obtains the asymptotic expansion for any compact set $K$ of positive capacity and computes the explicit forms of the first three coefficients of the expansion in terms of the logarithmic capacity of $K$, the coefficients agreeing with those conjectured in [3]. It is warned that these authors define the sausage for the Brownian motion $\tilde{B}_t = B_{2t}$ instead of $B_t$ so that to translate our results to their case one must replace $t$ by $2t$ in our formulae. The coefficients of our formulae of course agree with those obtained previously, whether it is a free or conditioned Brownian motion which forms the sausage, as readily ascertained by substituting $\alpha = \kappa/r^2$ in (1.2) and (1.3) and noting that the logarithmic capacity of the disc of radius $r$ equals $r^2$ (according to the normalization of logarithmic capacity in them). This suggests that our formulae in Theorems 1.1 and 1.2 would be extended to the sausage swept by any non-polar compact set $K$ in place of the disc if $r$ is replaced by the logarithmic capacity of $K$.

The higher dimensional case of free Brownian motion is treated by Spitzer [22], [8] and Le Gall [15] for a compact set and by [9] for a ball. The pinned cases are dealt with by Uhlenbeck and Beth [28] and McGillivray [18]. A more detailed account of the results for the expected volume (for dimensions $\geq 2$) obtained up to 1997 can be found in [3].

The function $N(t)$ expressing the leading terms in our theorems has already appeared in an analogous manner for the corresponding problem concerning the range of random walks (cf. [24]) (in which $N'(\lambda)$ is denoted by $W(\lambda)$). The expression by means of $N(t)$ makes possible a subtle comparison between the expected area of the Wiener sausage and the corresponding quantity to a random walk of mean zero: we can associate a certain natural 'radius', say $r_*$, of a lattice point of $\mathbf{Z}^2$ with the random walk ([25], Remark 6), and according to Corollary 1.1 of [24] and our Theorem 1.2 the expected number of sites visited by the walk in the first $n$ steps and the expected area of the Wiener sausage over

the internal $[0,n]$ coincide up to the error of $O(1)$ for the processes conditioned to return to the origin at the time $n$, provided that $r$ is chosen to be $r_*$, the variances of Brownian and random walk processes are the same and the fourth moments of the random walks exist.

The results obtained in this paper are quite parallel with those in [24], but the methods and the structures of the proofs are considerably different. In the random walk case we have simple expressions of the expected 'area' of the range of the walk for both free and conditional ones due to the discrete nature of the walk, which is not available to the Brownian case. In (cf. [24]) the Fourier analysis is the main tool, while in the present paper it plays a minor, though fundamental, role. We need a careful evaluation of certain integrals to find out the form of the second term as given in the formula of Theorem 1.2. Our derivation of it is directed by the result of [24], the agreement of asymptotic forms of various quantities for Brownian motion and Random walks being expected. The precise estimates of the first hitting time distributions for corresponding processes as obtained in [25] and [26] are used in both papers but the usage is much more essential for the present than in [25]. One can obtain an asymptotic estimate of the density of the first hitting time distribution valid uniformly with respect to starting positions and, with it, extend the result of Theorem 1.2 to the case when $|\mathbf{x}|/\sqrt{t} \to \infty$, which will be studied in another paper [27].

The following notation will be used: $a \wedge b = \min\{a,b\}$, $a \vee b = \max\{a,b\}$ $(a,b \in \mathbf{R})$; two dimensional points are denoted by bold face letters $\mathbf{z}, \mathbf{x}, \mathbf{y}$, $|\mathbf{z}|$ denotes the Euclidean length of $\mathbf{z}$; and $\mathbf{x} \cdot \mathbf{z}$ the Euclidian inner product of $\mathbf{x}$ and $\mathbf{z}$; for functions $g$ and $G$ of a variable $x$, $g(x) = O(G(x))$ means that there exists a constant $C$ such that $|g(x)| \le C|G(x)|$ whenever $x$ ranges over a specified set; the letters $C, C', C''$ etc. denotes constants whose values are not significant and may change in different places where they occur.

We prove Theorem 1.1 in Section 2 and Theorem 1.2 in Section 4. In Section 3 we give some results on the distribution of the first hitting time to a disc, which prepare for the analysis made in Section 4. In the last section we derive the asymptotic expansions associated with $N(t)$ as mentioned in Remark 1.

## 2 Proof of Theorem 1.1

We shall consider the Brownian motion started at $\mathbf{z}$ and denotes its law by $P_{\mathbf{z}}$. Let $\sigma^{(r)} = \sigma_{U(r)}$ be the first hitting time of Brownian motion $B_t$ to the disc $U(r)$ of radius $r$ and centered at the origin. Then

$$E[\mathrm{Area}(S_t^{(r)})] = \int_{\mathbf{R}^2} P_{\mathbf{z}}[\sigma^{(r)} < t] d\mathbf{z}.$$

Let $q(\mathbf{z};r)$ denote the density of the distribution of $\sigma^{(r)}$:

$$q(\mathbf{z},t;r) = \frac{d}{dt} P_{\mathbf{z}}[\sigma^{(r)} < t],$$

so that

$$\frac{d}{dt} E[\mathrm{Area}(S_t^{(r)})] = \int_{\mathbf{z}>r} q(\mathbf{z},t;r) d\mathbf{z}. \tag{2.1}$$

Let $p_t(\mathbf{z})$ denote the heat kernel on the plane.:

$$p_t(\mathbf{z}) = (2\pi t)^{-1} e^{-\mathbf{z}^2/2t}.$$

As is well-known we have

$$\int_0^\infty p_t(\mathbf{z})e^{-\lambda t}dt = \frac{1}{\pi}K_0(|\mathbf{z}|\sqrt{2\lambda})$$

(cf. [12], [6]), hence

$$\int_0^\infty q(\mathbf{z},t;r)e^{-\lambda t}dt = \frac{G_\lambda(0,|\mathbf{z}|)}{G_\lambda(0,r)} = \frac{K_0(|\mathbf{z}|\sqrt{2\lambda})}{K_0(r\sqrt{2\lambda})}, \tag{2.2}$$

where $G_\lambda$ denotes the resolvent kernel for the 2-dimensional Bessel process and $K_\nu$ the usual modified Bessel function of second kind of order $\nu$. Put

$$m(t;r) = \int_{|\mathbf{z}|>r} q(\mathbf{z},t;r)d\mathbf{z}.$$

From (2.1) and (2.2) we have

$$\begin{aligned}\int_0^\infty m(t;r)e^{-\lambda t}dt &= 2\pi\int_r^\infty \frac{K_0(u\sqrt{2\lambda})}{K_0(r\sqrt{2\lambda})}udu \\ &= \frac{2\pi r K_1(r\sqrt{2\lambda})}{K_0(r\sqrt{2\lambda})\sqrt{2\lambda}}, \end{aligned} \tag{2.3}$$

where we have applied the identity $(d/dz)[zK_1(z)] = -zK_0(z)$ for the second equality. From the scaling property of Brownian motion it follows that

$$m(t,r) = m(t/r^2;1)$$

and we suppose $r=1$ in what follows. By Laplace inversion

$$m(t,1) = \int_{-\infty}^\infty \frac{K_1(\sqrt{2iu})}{K_0(\sqrt{2iu})\sqrt{2iu}}\, e^{itu}du. \tag{2.4}$$

Throughout the paper $\lg z$ and $\sqrt{z}$ denote the principal branches in $-\pi < \arg z < \pi$ of the logarithm and the square root, respectively.

Although one can derive a leading term of $m(t;1)$ directly from (2.4) as in [26] (the proofs of Lemmas 4 and 5), here we use the formula

$$\int_0^\infty N(t)e^{-\lambda t}dt = \frac{1}{\lambda-1} - \frac{1}{\lambda\lg\lambda} \qquad (\lambda>0) \tag{2.5}$$

(cf. [11], page 196; also [7]), which somewhat simplifies the proof. Put

$$\varphi(z) = \frac{K_1(\sqrt{2z})}{K_0(\sqrt{2z})\sqrt{2z}} - \frac{1}{\kappa}\Big[\frac{1}{\kappa^{-1}z-1} - \frac{1}{\kappa^{-1}z\lg(\kappa^{-1}z)}\Big].$$

Then (2.3) is written as

$$\int_0^\infty m(t;1)e^{-\lambda t}dt = 2\pi\int_0^\infty N(\kappa t)e^{-\lambda t}dt + 2\pi\varphi(\lambda).$$

Accordingly (2.4) becomes

$$m(t,1) - 2\pi N(\kappa t) = \int_{-\infty}^\infty \varphi(iu)e^{itu}du. \tag{2.6}$$

Here (and in below) the trigonometric integral at infinity is improper. Note that $\varphi(z)$ is analytic on the slit domain $-\pi < \arg z < \pi$ since $K_0(\sqrt{z})$ has no zeros in it: the apparent singularity at $z = \kappa$ (i.e., that at $\lambda = 1$ on the right-hand side of (2.5)) is removable; also $\varphi(iu)$ tends to zero as $u \to \infty$ and is bounded about the origin as will be observed shortly.

The idea of the proof of the first formula in Theorem 1.1 would now be obvious. The asymptotic behavior of the Fourier integral on the right side of (2.6) for large values of $t$ depends on that of $\varphi(iu)$ near zero, provided that $\varphi(iu)$ behaves sufficiently regularly.

In view of the asymptotic formula

$$K_\nu(\sqrt{2z}) = \left(\frac{\pi}{2\sqrt{2z}}\right)^{1/2} e^{-\sqrt{2z}}\left(1 + \frac{\nu^2 - \frac{1}{4}}{2\sqrt{2z}} + O(|z|^{-1})\right) \qquad \text{as} \quad |z| \to \infty \tag{2.7}$$

$(-\pi < \arg z < \pi,\ \nu \geq 0)$ (cf. [14] (5.11.9)), we have

$$\varphi(iu) = \frac{1}{\sqrt{2iu}} - \frac{3}{4iu} + \frac{1}{iu\lg(iu/\kappa)} + R(u), \tag{2.8}$$

where $R(u) = O(|u|^{-3/2})$ with its derivatives $R^{(j)}(u) = O(|u|^{-3/2-j})$ as $|u| \to \infty$.

For the first formula of Theorem 1.1 it suffices to show that as $t \to \infty$

$$\int_{-\infty}^{\infty} \varphi(iu)w(u)e^{itu}du = \frac{4\pi}{t(\lg t)^3}(1 + o(1)), \tag{2.9}$$

where $w(u)$ is a smooth function that equals 1 in a neighborhood of the origin and vanishes outside a finite interval, for the $1 - w(u)$ part contributes to the integral at most $O(1/t^N)$ for any $N > 1$ so that it is negligible.

Put

$$g(z) = -\lg\left(\tfrac{1}{2}\sqrt{2z}\,\right) - \gamma = -2^{-1}\lg(\kappa^{-1}z). \tag{2.10}$$

By definition

$$K_0(z) = \sum_{k=0}^{\infty}\frac{(z/2)^{2k}}{(k!)^2}\left(\sum_{m=1}^{k}\frac{1}{m} - \gamma - \lg(\tfrac{1}{2}z)\right) \tag{2.11}$$

and

$$K_1(z) = \frac{1}{z} + \frac{z}{2}\sum_{k=0}^{\infty}\frac{(z/2)^{2k}}{k!(k+1)!}\left(\lg\frac{z}{2} + \gamma + \frac{1}{2(k+1)} - \sum_{m=1}^{k}\frac{1}{m}\right)$$

for $-\pi < \arg z < \pi$. It follows that for $-1/2 < u < 1/2$

$$K_0(\sqrt{2iu}\,) = g(iu) + 2^{-1}iu(g(iu) + 1) + R_1(u); \tag{2.12}$$

$$\frac{K_1(\sqrt{2iu}\,)}{\sqrt{2iu}} = \frac{1}{2iu} + \frac{1}{2}\left(-g(iu) + \frac{1}{2}\right) + R_2(u); \tag{2.13}$$

and

$$\frac{1}{K_0(\sqrt{2iu}\,)} = \frac{1}{g(iu)} + \frac{-iu}{2g(iu)}\left(1 + \frac{1}{g(iu)}\right) + R_3(u). \tag{2.14}$$

Here $R_1(u) = O(u^2)\times g(iu)$, $R_2(u) = O(u)\times g(iu)$ and $R_3(u) = O(u^2)/g(iu)$. Substitution of these together with an easy computation yields

$$\varphi(iu) = -\frac{1}{2} + \frac{1}{\kappa} - \frac{1}{[2g(iu)]^2} + R_4(u) \qquad (-1/2 < u < 1/2)$$

where $R_4(u) = O(u)$ with the derivatives $R_4'(u) = O(1), R_4^{(1+j)}(u) = O(1/u^j \lg|u|)$ $(j = 1, 2)$. For evaluation of the contribution of $R_4$ to the Fourier integral in (2.9) we split its range at $u = \pm 1/t$. The inner part plainly gives the bound $O(1/t^2)$, whereas for the outer part $\int_{|u|>1/t} R_4(u)w(u)e^{itu}du$ we repeat the integration by parts four times, which leads to the same bound of $O(1/t^2)$ (see Lemma 2.2 of [25] for more details). Hence

$$\int_{-\infty}^{\infty} \varphi(iu)w(u)e^{itu}du = \int_{-\infty}^{\infty} \frac{-e^{itu}}{[2g(iu)]^2}du + O\Big(\frac{1}{t^2}\Big).$$

We rewrite the integral on the right-hand side as

$$\frac{\kappa}{i}\int_{-i\infty}^{i\infty} \frac{-e^{\kappa tz}}{[\lg z]^2}dz$$

and apply the Cauchy integral theorem. The integral along the lower (resp. upper) half of the imaginary axis equals the one along the lower (resp. upper) side of the negative real axis in the positive (resp. negative) direction. Noting $\lg(-x \pm i0) = \lg x \pm i\pi$ for $x > 0$ we then find the foregoing integral equal to

$$\kappa\int_0^{\infty} \frac{-4\pi(\lg x)e^{-\kappa tx}}{[(\lg x)^2+\pi^2]^2}dx = \frac{4\pi}{t(\lg t)^3}(1+o(1))$$

(as $t \to \infty$). Thus the relation (2.9), and hence the first formula of Theorem 1.1, has been proved.

Consider the case $t \downarrow 0$. Here we go back to the original inversion integral in (2.4). From the asymptotic formula (2.7) we observe as before that the integrand of it involves the singular components $1/\sqrt{2iu}$ and $1/4iu$ that are not integrable at infinity and we evaluate the contributions of them separately. To this end we bring in a function $\psi(z)$ by

$$\psi(z) = \frac{K_1(\sqrt{2z})}{K_0(\sqrt{2z})\sqrt{2z}} - \frac{1}{\sqrt{2z}} - \frac{1}{4(z+1)},$$

so that $\psi(z) = O\Big(|z|^{-3/2}\Big)$ as $z \to \infty$ in the domain $-\pi < \arg z < \pi$ where $\psi(z)$ is regular. In addition we have $\psi(z) = o(1/z)$ if $z \to 0$ and apply the Cauchy integral theorem to see that the principal value integral $p.v. \int_{-\infty}^{\infty} \psi(iu)du$ vanishes, so that as $t \downarrow 0$

$$p.v. \int_{-\infty}^{\infty} \psi(iu)e^{itu}du = \int_{-\infty}^{\infty} \psi(iu)\Big[e^{itu}-1\Big]du = O(\sqrt{t}\,), \tag{2.15}$$

where the last equality may be verified by splitting the integral at $u = \pm 1/t$.

On the other hand

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2iu}}e^{itu}du = \int_0^{\infty} \frac{\cos u + \sin u}{\sqrt{u}}du = \sqrt{\frac{2\pi}{t}}, \tag{2.16}$$

and

$$\int_{-\infty}^{\infty} \frac{1}{4(iu+1)}e^{itu}du = \frac{1}{2}\int_0^{\infty} \frac{\cos tu + u\sin tu}{u^2+1}du = \frac{\pi}{2}e^{-t}. \tag{2.17}$$

From (2.15), (2.16) and (2.17) we find the formula as $t \downarrow 0$ as asserted in Theorem 1.1.

The proof of Theorem 1.1 is complete.

## 3 Preliminary results on the first hitting time to a disc

Here we give several results which prepare for estimation of the expected area of the sausages for the Brownian bridges. Recall that $q(\mathbf{z}, t; r)$ denotes the density of the distribution of $\sigma^{(r)}$ the first hitting time at the disc $U(r)$ of Brownian motion $B_t$ started at $\mathbf{z}$. The following result is proved in [26].

**Theorem 3.1** *Uniformly for $|\mathbf{z}| > r$, as $t \to \infty$*

$$q(\mathbf{z}, t; r) = \frac{\lg(\frac{1}{2}\kappa(\mathbf{z}/r)^2)}{(\lg(\kappa t/r^2))^2 t} e^{-\mathbf{z}^2/2t} + \begin{cases} \dfrac{2\gamma \lg(t/\mathbf{z}^2)}{t(\lg t)^3} + O\Big(\dfrac{1}{t(\lg t)^3}\Big) & \text{for} \quad \mathbf{z}^2 < t, \\ O\Big(\dfrac{1 + [\lg(\mathbf{z}^2/t)]^2}{\mathbf{z}^2(\lg t)^3}\Big) & \text{for} \quad \mathbf{z}^2 \geq t. \end{cases} \tag{3.1}$$

Let $\sigma = \sigma^{(1)}$ and $q(\mathbf{z}, t) = q(\mathbf{z}, t; 1)$. We shall use the estimate of this theorem in the following slightly reduced form.

**Corollary 3.1** *Uniformly for $|\mathbf{z}| > 1$, as $t \to \infty$*

$$\begin{aligned} q(\mathbf{z}, t) &= \frac{\lg(\frac{1}{2}\kappa(\mathbf{z}/r)^2)}{(\lg(\kappa t/r^2))^2} 2\pi p_t(\mathbf{z}) + O\Big(\frac{|\lg[(t/\mathbf{z}^2) \vee 2]|}{t(\lg t)^3}\Big) && \text{for} \quad |\mathbf{z}|^2 < 4t\lg(\lg t), \\ &= O\Big(\frac{1}{t(\lg t)^3}\Big) && \text{for} \quad |\mathbf{z}|^2 \geq 4t\lg(\lg t). \end{aligned}$$

*Proof.* Using the inequality $\lg(\mathbf{z}^2/t) < \mathbf{z}^2/t$ for $\mathbf{z}^2 > t$ the assertion is immediate from Theorem 3.1. □

The following crude bound is often useful for dealing with the case $|\mathbf{z}| > \sqrt{6t \lg \lg t}$.

**Lemma 3.1** *There is a constant $c > 0$ such that for all $|\mathbf{z}| > 1$ and $t > 1$,*

$$q(\mathbf{z}, t) \leq c\, p_{t+1}(\mathbf{z}). \tag{3.2}$$

*Proof.* For any unit vector $\xi$, $p_s(\xi)$ depends only on $s$ and we see

$$p_t(\mathbf{z}) = \int_0^t q(\mathbf{z}, t-s) p_s(\xi) ds.$$

Hence

$$p_{t+1}(\mathbf{z}) \geq \int_0^1 q(\mathbf{z}, t+1-s) \frac{e^{-1/2s}}{2\pi s} ds$$

and, applying the parabolic Harnack inequality (cf. [20]) which implies that $q(\mathbf{z}, t) \leq Cq(\mathbf{z}, t+1-s)$ for $0 \leq s \leq 1, |\mathbf{z}| > 2$, we obtain the inequality of the lemma. (Note that for $|\mathbf{z}| < 2$, a better estimate is given in Corollary 3.1.) □

Remark 3. On the right-hand side of (3.2) we can replace $p_{t+1}(\mathbf{z})$ by $p_t(\mathbf{z})$ for $|\mathbf{z}| < Mt$, provided that at the same time $c$ is replaced by a constant $c_M$ which may depend on $M$. It is warned that the constant $c_M$ actually depends on $M$ and in fact the ratio $q(\mathbf{z}, t)/p_t(\mathbf{z})$ tends to infinity whenever $|\mathbf{z}|/t \to \infty$, $t \to \infty$.

**Lemma 3.2** *For all $|\mathbf{z}| > 1$ and $t > 0$,*

$$P_{\mathbf{z}}[\sigma < t] \le \sqrt{\frac{2e}{\pi}} \left( \frac{\sqrt{t}}{|\mathbf{z}| - 1} \wedge 1 \right) e^{-(|\mathbf{z}|-1)^2/2t}. \tag{3.3}$$

*Proof.* The relation follows from the one dimensional result that if $B_t^{(2)}$ denotes the vertical component of $B_t$, $P_{(0,|\mathbf{z}|)}[B_s^{(2)} > 1$ for $0 < s < t] = 2(2\pi t)^{-1/2} \int_{(|\mathbf{z}|-1)}^{\infty} e^{-u^2/2t} du$. □

**Lemma 3.3** *There exists $c_1 > 0$ such that for $|\mathbf{z}| > 2$ and $t \ge 1$,*

$$0 \le \frac{d}{dt} E_{\mathbf{z}}[B_\sigma \cdot \mathbf{z}; \sigma < t] \le c_1 \frac{\mathbf{z}^2}{t} p_{t+1}(\mathbf{z}).$$

*Proof.* The expectation $E_{\mathbf{z}}[B_\sigma \cdot \mathbf{z}; \sigma < t]$ is a radial function of $\mathbf{z}$ and we may suppose $\mathbf{z}$ is on the upper vertical axis. Let $\tau_0$ be the first exit time from $\mathbf{H} \setminus U(1)$, where $\mathbf{H}$ denotes the upper half plane. Let $B_t^{(1)}$ and $B_t^{(2)}$ be the horizontal and vertical components, respectively, of $B_t$. Then by symmetry, for $\mathbf{z} = (0, y), y > 1$,

$$\frac{d}{dt} E_{(0,y)}[B_\sigma \cdot \mathbf{z}; \sigma < t] = y \frac{d}{dt} E_{(0,y)}[B_{\tau_0}^{(2)}; B_{\tau_0} \in \mathbf{H}, \tau_0 < t].$$

The derivative on the right-hand side is expressed as the integral of the vertical component of $\xi \in \mathbf{H} \cap \partial U(1)$ by $P_{(0,y)}[B_{\tau_0} \in d\xi, \tau_0 \in dt]/dt$. It therefore suffices to show that

$$\frac{d}{dt} P_{(0,y)}[B_{\tau_0} \in \mathbf{H}, \tau_0 < t] \le C \frac{y}{t^2} e^{-y^2/2t}. \tag{3.4}$$

For verification let $D = \mathbf{H} - (0,1) = \{(x, y-1) : (x,y) \in \mathbf{H}\}$. Then, on denoting by $\tau_D = \tau(D)$ the first exit time from $D$,

$$\begin{aligned} & P_{(0,y)}[\tau_D \in dt, \, -1 < B_{\tau(D)}^{(1)} < 1] \\ & \ge \int_{|\xi|=1, \xi \in \mathbf{H}} \int_0^t ds \frac{d}{ds} P_{(0,y)}[B_{\tau_0} \in d\xi, \tau_0 \le s] P_\xi[B_{\tau(D)}^{(1)} \in (-1,1), \tau_D + s \in dt]. \end{aligned}$$

Since

$$P_{(x,y)}[\tau_D \in dt, B_{\tau(D)}^{(1)} \in du] = p_t((x-u, y+1)) \frac{y+1}{t} dt du,$$

we obtain

$$\begin{aligned} 2p_t((0, y+1)) \frac{y+1}{t} &\ge \int_{|\xi|=1, \xi \in \mathbf{H}} \int_0^t \frac{d}{ds} P_{(0,y)}[B_{\tau_0} \in d\xi, \tau_0 \le s] \frac{2p_{t-s}(\xi + (0,1))}{t-s} ds \\ &\ge c \int_{t-1/2}^{t-1/4} \frac{d}{ds} P_{(0,y)}[B_{\tau_0} \in \mathbf{H}, \tau_0 \le s] ds. \end{aligned}$$

for some universal constant $c > 0$. Thus the bound of (3.4) follows in view of the parabolic Harnack inequality as before.

# 4 Wiener sausages for Brownian bridges

For $\mathbf{x} \in \mathbf{R}^2$, put

$$F(t, \mathbf{x}; r) = \int_{|\mathbf{z}| \geq r} d\mathbf{z} \int_0^t \int_{|\xi|=r} P_{\mathbf{z}}[\sigma_r \in ds, B_{\sigma_r} \in d\xi] p_{t-s}(\mathbf{x} - \mathbf{z} - \xi).$$

Then

$$E[\mathrm{Area}(S_t^{(r)}) \mid B_t = \mathbf{x}] = \frac{1}{p_t(\mathbf{x})} F(t, \mathbf{x}; r) + \pi r^2. \tag{4.1}$$

From the scaling property of Brownian motion it follows that

$$F(t, \mathbf{x}; r) = F(t/r^2, \mathbf{x}/r; 1)$$

and we have only to consider the case $r = 1$ as in the proof of Theorem 1.1. Put

$$F_0(t, \mathbf{x}) = \int_{|\mathbf{z}| \geq 1} d\mathbf{z} \int_0^t P_{\mathbf{z}}[\sigma \in ds] p_{t-s}(\mathbf{z} - \mathbf{x}) = \int_{|\mathbf{z}| \geq 1} d\mathbf{z} \int_0^t q(\mathbf{z}, s) p_{t-s}(\mathbf{z} - \mathbf{x}) ds.$$

We first consider the case $\mathbf{x} = 0$ in Subsection 4.1. The general case, dealt with in Subsection 4.2, is based on the proof of this special case but need additional estimations of a certain integral that are involved and partly delicate.

## 4.1 The case $\mathbf{x} = 0$

**Lemma 4.1** *Let $F$ and $F_0$ be as above. Then $F(t, 0; 1) = F_0(t, 0) + O(1/t)$.*

*Proof.* Let $a$ be a constant larger than 1. In this proof it may be arbitrarily fixed (eg. $a = 2$; in the case $\mathbf{x} \neq 0$ treated later we shall take $a = \mathbf{x}^2$). We split the range of integration by the surfaces $s = t - a$ and $|\mathbf{z}| = \sqrt{4(t-s)\lg(t-s)}$ put

$$\begin{aligned}
D_0 &= D_0(a) = \{(s, \mathbf{z}) : t - a < s \leq t, |\mathbf{z}| > 1\} \\
D_> &= D_>(a) = \{(s, \mathbf{z}) : 0 < s \leq t - a, |\mathbf{z}| > \sqrt{4(t-s)\lg(t-s)},\ |\mathbf{z}| > 1\}, \\
D_< &= D_<(a) = \{(s, \mathbf{z}) : 0 < s \leq t - a, |\mathbf{z}| \leq \sqrt{4(t-s)\lg(t-s)},\ |\mathbf{z}| > 1\}.
\end{aligned}$$

Accordingly we break the rest of the proof into three parts. Some estimates obtained below are more accurate than necessary for the poof of the present lemma, but they are needed in the proof of Lemma 4.4 essential for our proof of Theorem 1.2.

*Part 1* : $D_0$. The contribution to the integrals defining $F$ and $F_0$ from $D_0$ is $O(1/t(\lg t)^2)$. Indeed on the one hand we can use the bound (3.2) for the integration w.r.t. $\mathbf{z}$ over the range $|\mathbf{z}| > \sqrt{4t \lg \lg t}$, in which $p_{t+1}(\mathbf{z}) = O(1/t(\lg t)^2)$. On the other hand, on applying Corollary 3.1 the contribution of the other part is at most a constant multiple of

$$\begin{aligned}
&\int_{t-a}^t \frac{ds}{s(\lg s)^2} \int_{1 < |\mathbf{z}| < \sqrt{4t \lg \lg t}} (\lg |\mathbf{z}|) \sup_{|\xi|=1} p_{t-s}(\mathbf{z} - \xi) d\mathbf{z} \\
&\leq \frac{C}{t(\lg t)^2}.
\end{aligned} \tag{4.2}$$

*Part 2* $:D_>$. For $(s,\mathbf{z}) \in D_>$, according to Corollary 3.1 $q(\mathbf{z},s) \le C/s(\lg s)^3$ if $a < s < t/2$ and $q(\mathbf{z},s) \le C/s\lg s$ if $s > t/2$. Hence both the contributions of $D_>$ to $F$ and $F_0$ are dominated from above by a constant multiple of

$$\begin{aligned}
&\int_4^{t-a}\left[\frac{\mathbf{1}(s\le t/2)}{s(\lg s)^3}+\frac{\mathbf{1}(s>t/2)}{s\lg s}\right]ds\int_{|\mathbf{z}|>\sqrt{4(t-s)\lg(t-s)}}\sup_{|\xi|=1} p_{t-s}(\mathbf{z}-\xi)d\mathbf{z}\\
&\le \frac{C'}{t^2}+C'\int_{t/2}^{t-a}\frac{ds}{(s\lg s)(t-s)^2}\\
&= O\Big(\frac{1}{t\lg t}\Big).
\end{aligned}$$

On using (3.3) the integral on $0 < s \le 4$ is readily evaluated to be at most $O(1/t^2)$.

*Part 3* $:D_<$. Observe that if $(s,\mathbf{z}) \in D_<$, then

$$\begin{aligned}
p_{t-s}(\mathbf{z}-\xi)-p_{t-s}(\mathbf{z}) &= p_{t-s}(\mathbf{z})(e^{[\mathbf{z}\cdot\xi-\frac12]/(t-s)}-1)\\
&= p_{t-s}(\mathbf{z})\frac{\mathbf{z}\cdot\xi-\frac12}{t-s}+\eta_t(s,\mathbf{z},\xi)
\end{aligned}\tag{4.3}$$

with

$$\eta_t(s,\mathbf{z},\xi) = p_{t-s}(\mathbf{z})\times O\Big(\frac{\mathbf{z}^2+1}{(t-s)^2}\Big).$$

The integral for the difference $F(t,0;1)-F_0(t,0)$ restricted to

$$D_<^0 := \{(s,\mathbf{z}) \in D_< : 0 < s \le 4\}$$

is at most $O(1/t^2)$ in view of (3.3). Thus we may restrict the integral to the range $D_<\backslash D_<^0$.

Consider the contribution of the first term on the right side of (4.3). Applying Lemma 3.3 with the help of Corollary 3.1 (for $|\mathbf{z}| < 2$) we deduce that

$$\begin{aligned}
&\int_{D_<\backslash D_<^0} dsd\mathbf{z}\left|\int_{|\xi|=1} p_{t-s}(\mathbf{z})\frac{\mathbf{z}\cdot\xi}{t-s}\cdot\frac{P_{\mathbf{z}}[\sigma\in ds, B_\sigma\in d\xi]}{ds}\right|\\
&\le C\int_4^{t-a}ds\int_{|\mathbf{z}|>2}\frac{p_{t-s}(\mathbf{z})\mathbf{z}^2p_{s+1}(\mathbf{z})}{(t-s)s}d\mathbf{z}+C\int_4^{t-a}\frac{ds}{(t-s)^2s(\lg s)^2}\le\frac{C'}{t}.
\end{aligned}\tag{4.4}$$

Here we have applied the trivial bound $p_{t-s}(\mathbf{z}) < 1/(t-s)$ (for the integral on $s < t/2$) as well as $p_{s+1}(\mathbf{z}) \le 1/s$ (for that on $s > t/2$). The part involving the term $1/2$ is evaluated to be $O(1/t)$ by dominating $q(\mathbf{z},s)$ by $Cp_{s+1}(\mathbf{z})$ for $s < t/2$ and by $C/s\lg s$ for $s > t/2$.

For the proof of the lemma it now suffices to show that

$$\int_{D_<\backslash D_<^0} dsd\mathbf{z}\int_{|\xi|=1}\eta_t(s,\mathbf{z},\xi)\cdot\frac{P_{\mathbf{z}}[\sigma\in ds, B_\sigma\in d\xi]}{ds}=O\Big(\frac{1}{t\lg t}\Big).\tag{4.5}$$

To this end we further split the region $D_<$ by the surfaces $s = t/2$ and $|\mathbf{z}| = 1+\sqrt{4s\lg\lg s}$ $(4 < s < t/2)$. Denote by $J^{\mathrm{RH}}(a)$, $J_{\mathrm{LH}}^{(>)}$, $J_{\mathrm{LH}}^{(<)}$ the double integrals on the regions

$$\begin{aligned}
&D_<^{\mathrm{RH}}(a) := \{(s,\mathbf{z}) \in D_< : t/2 \le s < t-a\},\\
&D_{\mathrm{LH}}^{(>)} := \{(s,\mathbf{z}) \in D_< : 4 < s < t/2, |\mathbf{z}| \ge 1+\sqrt{4s\lg\lg s}\,\},\\
&D_{\mathrm{LH}}^{(<)} := \{(s,\mathbf{z}) \in D_< : 4 < s < t/2, |\mathbf{z}| < 1+\sqrt{4s\lg\lg s}\,\},
\end{aligned}$$

respectively. Employing Corollary 3.1 we obtain

$$\begin{aligned}
J^{\rm RH}(a) &< C\int_{t/2}^{t-a}\frac{ds}{s\lg s}\int_{|\mathbf{z}|<\sqrt{4(t-s)\lg(t-s)}}\frac{\mathbf{z}^2}{(t-s)^2}p_{t-s}(\mathbf{z})d\mathbf{z}\\
&< \frac{C'}{t\lg t}\int_{t/2}^{t-a}\frac{ds}{t-s}=O\Big(\frac{1}{t}\Big),
\end{aligned}\tag{4.6}$$

and, similarly but dominating $p_{t-s}(\mathbf{z})$ simply by $1/(t-s)$,

$$\begin{aligned}
J^{(<)}_{\rm LH} &< C\int_4^{t/2}\frac{ds}{(t-s)^3}\int_{|\mathbf{z}|<1+\sqrt{4s\lg\lg s}}\frac{\mathbf{z}^2}{\lg s}p_s(\mathbf{z})d\mathbf{z}\\
&< \frac{C'}{t^3}\int_4^{t/2}\frac{s}{\lg s}ds=O\Big(\frac{1}{t\lg t}\Big).
\end{aligned}\tag{4.7}$$

With the help of (3.2) and (3.1) as well as (4.3) we deduce that

$$\begin{aligned}
J^{(>)}_{\rm LH} &< C\int_4^{t/2}ds\bigg[\int_{|\mathbf{z}|>1+\sqrt{6s\lg\lg s}}p_{s+1}(\mathbf{z})\frac{|\mathbf{z}|^2p_{t-s}(\mathbf{z})}{(t-s)^2}d\mathbf{z}\\
&\qquad +\int_{|\mathbf{z}|>1+\sqrt{4s\lg\lg s}}\Big\{\frac{p_s(\mathbf{z})}{\lg s}+\frac{(\lg\lg s)^2}{\mathbf{z}^2(\lg s)^3}\Big\}\frac{\mathbf{z}^2p_{t-s}(\mathbf{z})}{(t-s)^2}d\mathbf{z}\bigg]\\
&< C'\int_4^{t/2}ds\bigg[\frac{1}{(\lg s)^2t^2}+\int_{|\mathbf{z}|>1+\sqrt{4s\lg\lg s}}\frac{\mathbf{z}^2p_s(\mathbf{z})d\mathbf{z}}{t^3\lg s}+\int_{|\mathbf{z}|>1+\sqrt{6s\lg\lg s}}\frac{\mathbf{z}^2p_s(\mathbf{z})d\mathbf{z}}{t^3}\bigg]\\
&\le \frac{C''}{t(\lg t)^2}.
\end{aligned}$$

Thus (4.5) has been proved. The proof of the lemma is complete. □

**Lemma 4.2**

$$F_0(t,0)=N(\kappa t)-\pi p_t(1)+O(1/t(\lg t)^2).$$

*Proof.* The Laplace transform of $F_0$ is given by

$$\begin{aligned}
\int_0^\infty F_0(t,0)e^{-\lambda t}dt &= \int_0^\infty e^{-\lambda t}dt\int_{|\mathbf{z}|\ge1}d\mathbf{z}\int_0^t P_{\mathbf{z}}[\sigma\in ds]p_{t-s}(\mathbf{z})\\
&= \frac{1}{\pi}\int_{|\mathbf{z}|\ge1}\Big[K_0(|\mathbf{z}|\sqrt{2\lambda})\Big]^2d\mathbf{z}\,\frac{1}{K_0(\sqrt{2\lambda})}
\end{aligned}$$

(cf. [12] Sect.7.2). Fortunately we have the identity

$$rK_0^2(ra)=\frac{1}{2}\frac{d}{dr}\Big[r^2\Big(K_0^2(ra)-K_1^2(ra)\Big)\Big]$$

for any constant $a>0$, so that

$$\int_0^\infty F_0(t,0)e^{-\lambda t}dt=-K_0(\sqrt{2\lambda})+\frac{[K_1(\sqrt{2\lambda})]^2}{K_0(\sqrt{2\lambda})}.$$

Hence

$$F_0(t,0)=-\pi p_t(1)+\frac{1}{2\pi}\int_{-\infty}^\infty\frac{[K_1(\sqrt{2iu})]^2}{K_0(\sqrt{2iu})}e^{itu}du.$$

For evaluation of the last integral we can proceed as in the preceding section. Put

$$E(z) = \frac{[K_1(\sqrt{2z})]^2}{K_0(\sqrt{2z})} - \frac{1}{\kappa}\Big[\frac{1}{\kappa^{-1}z - 1} - \frac{1}{\kappa^{-1}z\lg(\kappa^{-1}z)}\Big].$$

Then for $-1/2 < u < 1/2$, $E(iu) = -1 + \frac{1}{\kappa} + [4g(iu)]^{-1} - [2g(iu)]^{-2} + R(u)$ with $R(u) = O(u)$ and as before (see (2.5) and the ensuing discussion up to (2.6)) we have

$$F_0(t,0) - N(\kappa t) + \pi p_t(1) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(iu)e^{itu}du = O\Big(\frac{1}{t(\lg t)^2}\Big).$$

The proof of the lemma is complete. □

Combining Lemmas 4.1 and 4.2 with (4.1) leads to the assertion of Theorem 1.2 with $\mathbf{x} = 0$.

## 4.2 The case $\mathbf{x} \neq 0$.

We first extend Lemma 4.1.

**Lemma 4.3** *For each $M \geq 1$ uniformly for $|\mathbf{x}| < M\sqrt{t}$, as $t \to \infty$*

$$F(t,\mathbf{x};1) = F_0(t,\mathbf{x}) + O(1/t).$$

*Proof.* First of all we verify that

$$\int_{|\mathbf{x}-\mathbf{z}|<1, |\mathbf{z}|>1} d\mathbf{z}\int_{t-1}^{t} P_{\mathbf{z}}[\sigma \in ds, B_\sigma \in d\xi]p_{t-s}(\mathbf{z}-\mathbf{x}-\xi) \leq \frac{C}{t\lg t}. \tag{4.8}$$

For verification we use the inequality

$$P_{\mathbf{z}}[\sigma \in ds, B_\sigma \in d\xi] \leq \int d\mathbf{y}p_1(\mathbf{y}-\mathbf{z})P_{\mathbf{y}}[\sigma + 1 \in ds, B_\sigma \in d\xi].$$

If $|\mathbf{x}-\mathbf{z}| < 1$, $p_1(\mathbf{y}-\mathbf{z})$ is dominated by $e^{-(\mathbf{y}-\mathbf{x})^2/4}$. Hence, performing the integration by $\mathbf{z}$ first, we see that the repeated integral in (4.8) is at most $\int_{t-1}^{t} ds\int e^{-(\mathbf{y}-\mathbf{x})^2/4}q(\mathbf{y}, s-1)d\mathbf{y}$, which is evaluated to be $O(1/t\lg t)$ according to Corollary 3.1.

The rest of the proof proceeds in parallel with that of Lemma 4.1. Of course we must replace $p_{t-s}(\mathbf{z}-\xi)$ and $p_{t-s}(\mathbf{z})$ by $p_{t-s}(\mathbf{z}-\mathbf{x}-\xi)$ and by $p_{t-s}(\mathbf{z}-\mathbf{x})$, respectively. We accordingly modify the definition of $D_<$ as

$$D_< = \{(s,\mathbf{z}) : 0 < s < t-a, |\mathbf{z}-\mathbf{x}| \leq \sqrt{4(t-s)\lg(t-s)}\}$$

and $D_>$ analogously. Then Parts 1 and 2 are treated without any change except that for Part 1 we use (4.8). For Part 3 we consider

$$\begin{aligned} p_{t-s}(\mathbf{z}-\mathbf{x}-\xi) - p_{t-s}(\mathbf{z}-\mathbf{x}) &= p_{t-s}(\mathbf{z}-\mathbf{x})(e^{[(\mathbf{z}-\mathbf{x})\cdot\xi - \frac{1}{2}]/(t-s)} - 1) \\ &= p_{t-s}(\mathbf{z}-\mathbf{x})\frac{(\mathbf{z}-\mathbf{x})\cdot\xi - \frac{1}{2}}{t-s} + \eta_{t,\mathbf{x}}(s,\mathbf{z},\xi) \end{aligned} \tag{4.9}$$

with

$$\eta_{t,\mathbf{x}}(s,\mathbf{z},\xi) = p_{t-s}(\mathbf{z}-\mathbf{x}) \times O\Big(\frac{|\mathbf{z}-\mathbf{x}|^2+1}{(t-s)^2}\Big)$$

in place of (4.3). In the first term on the last member of (4.9) $\mathbf{x}\cdot\xi$ may be replaced by $(\mathbf{x}\cdot\mathbf{z})(\mathbf{z}\cdot\xi)/\mathbf{z}^2$ since by symmetry the component of $\xi$ perpendicular to $\mathbf{z}$ vanishes after integration. It follows that $(\mathbf{z}-\mathbf{x})\cdot\xi$ can be replaced by $(\mathbf{z}\cdot\xi)\mathbf{z}\cdot(\mathbf{z}-\mathbf{x})/|\mathbf{z}|^2$. Keeping this in mind and employing (3.3) together with Lemma 3.3 and Corollary 3.1, we deduce that

$$
\begin{aligned}
&\int_{D_<} dsd\mathbf{z}\bigg|\int_{|\xi|=1} p_{t-s}(\mathbf{z}-\mathbf{x})\frac{(\mathbf{z}-\mathbf{x})\cdot\xi-\frac{1}{2}}{t-s}\cdot\frac{P_{\mathbf{z}}[\sigma\in ds, B_\sigma\in d\xi]}{ds}\bigg|\\
&\leq \frac{C}{t}+\int_a^{t-a}ds\int_{|\mathbf{z}|\geq 2}\frac{|\mathbf{z}\cdot(\mathbf{z}-\mathbf{x})|}{(t-s)s}\cdot p_{t-s}(\mathbf{z}-\mathbf{x})p_s(\mathbf{z})d\mathbf{z}\\
&\qquad +C\int_{|\mathbf{z}|<2}d\mathbf{z}\int_a^{t-a}|\mathbf{z}-\mathbf{x}|\,e^{-|\mathbf{z}-\mathbf{x}|^2/2(t-s)}\frac{ds}{(t-s)^2 s(\lg s)^2}\\
&\leq \frac{C'}{t}.
\end{aligned}\tag{4.10}
$$

Here the last inequality is due to the observation that the first integral in the middle member restricted on the interval $s\in[a,t/2]$ is at most a constant multiple of

$$
\frac{1}{t^2}\int_a^{t/2}ds\int_{\mathbf{R}^2}\frac{\mathbf{z}^2+|\mathbf{z}||\mathbf{x}|}{s}p_s(\mathbf{z})d\mathbf{z}\leq\frac{1}{t}+\frac{|\mathbf{x}|}{t^2}\int_0^{t/2}\frac{1}{\sqrt{s}}\,ds=O\Big(\frac{1}{t}\Big)
$$

and similarly for the other interval.

The rest of the proof is similarly dealt with and hence omitted. □

**Lemma 4.4** *Foe each $M\geq 1$ uniformly for $|\mathbf{x}|<M\sqrt{t}$, as $t\to\infty$*

$$
\begin{aligned}
F_0(t,\mathbf{x})-F_0(t,0) \;&=\; 2\pi\Big[p_t(\mathbf{x})-p_t(0)\Big]\frac{t}{\lg t}+\frac{\mathbf{x}^2}{2t(\lg t)^2}\Big[\lg\Big(\frac{t}{\mathbf{x}^2\vee 1}\Big)+O(1)\Big]\\
&\qquad +O\Big(\frac{1}{t\lg t}\Big).
\end{aligned}
$$

The right-hand side of the formula of this lemma is suggested by that of Theorem 1.2 and the latter in turn is by the corresponding one in [24] for random walks. The course of proof given below is often steered by the asserted formula.

*Proof of Lemma 4.4.* We suppose $|\mathbf{x}|<\sqrt{t}/2$ for simplicity of the description of the proof (otherwise one may take $a=\mathbf{x}^2/2M^2$ instead of $a=\mathbf{x}^2$ in below). Suppose also $|\mathbf{x}|\geq 2$, the same proof of Lemma 4.1 being applicable with little alteration if $|\mathbf{x}|<2$ . We make computations analogous to those in the proof of Lemma 4.1. We substitute $a=\mathbf{x}^2$ in the definitions of $D_0, D_>$ and $D_<$ given therein and denote the resulting regions by $D_0(\mathbf{x}^2)$, $D_>(\mathbf{x}^2)$ and $D_<(\mathbf{x}^2)$, respectively; denote the contributions from them to the difference $F_0(t,\mathbf{x})-F_0(t,0)$ by $I[D_0(\mathbf{x}^2)], I[D_>(\mathbf{x}^2)]$ and $I[D_<(\mathbf{x}^2)]$, respectively.

For the estimate given in Part 2 of the proof of Lemma 4.1 where the integral over $D_>(a)$ is dealt with we have the same bound $O(1/t\lg t)$, namely

$$
I[D_>(\mathbf{x}^2)]=O(1/t\lg t),\tag{4.11}
$$

since $|\mathbf{z}-\mathbf{x}|\geq|\mathbf{z}|-|\mathbf{x}|\geq\sqrt{3t\lg t}$ if $t/2<s<t-\mathbf{x}^2$ and $(s,\mathbf{z})\in D_>(\mathbf{x}^2)$ (the integral on $0<s<t/2$ is dealt with in the same way as before).

The rest of the proof is somewhat involved and broken into three parts **1** to **3**.

**1.** *Estimation of* $I[D_0(\mathbf{x}^2)]$. There exists a constant $C$ such that for $|\mathbf{x}| > 1$,

$$\int_0^{\mathbf{x}^2} du \int_{\mathbf{R}^2} \Big| \lg \mathbf{z}^2 - \lg |\mathbf{z}-\mathbf{x}|^2 \Big| p_u(\mathbf{z}) d\mathbf{z} < C\mathbf{x}^2 \tag{4.12}$$

as is proved shortly. For the integral that gives $I[D_0(\mathbf{x}^2)]$ we may restrict $(s, \mathbf{z})$ to $\{4 < t-s < \mathbf{x}^2, |\mathbf{z}| < \sqrt{4s \lg\lg s}\}$, the contribution of the remainder being at most $O(\mathbf{x}^2/t(\lg t)^2)$ as discussed before. In view of this as well as of (4.12) we may write it as

$$\int_{t-\mathbf{x}^2}^{t-4} ds \int_{|\mathbf{z}|<\sqrt{4s\lg\lg s}} \frac{2\pi p_s(\mathbf{z})}{(\lg s)^2} \Big[ (\lg|\mathbf{z}-\mathbf{x}|^2) p_{t-s}(\mathbf{z}-\mathbf{x}) - (\lg \mathbf{z}^2) p_{t-s}(\mathbf{z}) \Big] d\mathbf{z}$$
$$+ O\Big(\frac{\mathbf{x}^2}{t(\lg t)^2}\Big).$$

Substituting from the identities $\lg|\mathbf{z}-\mathbf{x}|^2 = \lg(t-s) + \lg[|\mathbf{z}-\mathbf{x}|^2/(t-s)]$ and $\lg \mathbf{z}^2 = \lg(t-s) + \lg[\mathbf{z}^2/(t-s)]$ and noting that $\int \lg(\mathbf{z}^2/(t-s)) p_{t-s}(\mathbf{z}) d\mathbf{z}$ equals $\int (\lg \mathbf{z}^2) p_1(\mathbf{z}) d\mathbf{z}$, a finite constant we find that

$$I[D_0(\mathbf{x}^2)] = 2\pi \Big[ p_t(\mathbf{x}) - p_t(0) \Big] \int_{t-\mathbf{x}^2}^{t-4} \frac{\lg(t-s) ds}{(\lg s)^2} + O\Big(\frac{\mathbf{x}^2}{t(\lg t)^2}\Big). \tag{4.13}$$

This is not negligible and to contribute to the leading term on the right-hand side of the formula of the lemma.

Proof of (4.12). If the range of integration is restricted to $|\mathbf{z}| \wedge |\mathbf{z}-\mathbf{x}| > |\mathbf{x}|/2$, then $\Big| \lg \mathbf{z}^2 - \lg|\mathbf{z}-\mathbf{x}|^2 \Big|$ is uniformly bounded and the corresponding (repeated) integral is obviously bounded by a positive multiple of $\mathbf{x}^2$. The integral corresponding to $\{|\mathbf{x}-\mathbf{z}| \le |\mathbf{x}|/2\}$ can be readily evaluated to be $O(\mathbf{x}^2)$. For the range $|\mathbf{z}| \le |\mathbf{x}|/2$ it suffices to show that

$$\frac{1}{\mathbf{x}^2} \int_0^{\mathbf{x}^2} \frac{du}{u} \int_{|\mathbf{z}|<|\mathbf{x}|} \Big( \lg \frac{|\mathbf{x}|}{|\mathbf{z}|} \Big) e^{-|\mathbf{z}|^2/2u} d\mathbf{z} < C'. \tag{4.14}$$

Changing the variables of integration according to $s = u/\mathbf{x}^2$ and $\mathbf{y} = \mathbf{z}/|\mathbf{x}|$ transforms the left-hand side of (4.14) to $\int_0^1 s^{-1} ds \int_{|\mathbf{y}|<1} (\lg|\mathbf{y}|) e^{-\mathbf{y}^2/2s} d\mathbf{y}$, which is independent of $x$. By another change of variables this last integral is further transformed to

$$\int_1^\infty \frac{ds}{s^2} \int_{|\mathbf{y}|<\sqrt{s}} \Big( \lg \frac{\sqrt{s}}{|\mathbf{y}|} \Big) e^{-\mathbf{y}^2/2} d\mathbf{y},$$

which is certainly finite. Hence we have (4.14).

**2.** *Estimation of* $I[D_<(\mathbf{x}^2)]$. For $(s, \mathbf{z}) \in D_<(\mathbf{x}^2)$, instead of (4.3) we have

$$\begin{aligned} p_{t-s}(\mathbf{z}-\mathbf{x}) - p_{t-s}(\mathbf{z}) &= p_{t-s}(\mathbf{z})(e^{[\mathbf{z}\cdot\mathbf{x} - \frac{1}{2}\mathbf{x}^2]/(t-s)} - 1) \\ &= p_{t-s}(\mathbf{z}) \Bigg[ \frac{\mathbf{z}\cdot\mathbf{x} - \frac{1}{2}\mathbf{x}^2}{t-s} + \frac{(\mathbf{z}\cdot\mathbf{x})^2}{2(t-s)^2} - \frac{(\mathbf{z}\cdot\mathbf{x})\mathbf{x}^2}{2(t-s)^2} \Bigg] \\ &\quad + \eta_{t,\mathbf{x}}(s, \mathbf{z}) \end{aligned} \tag{4.15}$$

with

$$\eta_{t,\mathbf{x}}(s, \mathbf{z}) = p_{t-s}(\mathbf{z}) \frac{\mathbf{x}^2}{t-s} \times O\Big( \frac{\mathbf{x}^2}{t-s} + \frac{|\mathbf{z}|^4}{(t-s)^2} \Big).$$

Since the double integral of the terms that are linear in $\mathbf{z}$ inside the big square brackets in (4.15) vanishes by skew symmetry, the substantial part of $p_{t-s}(\mathbf{z}-\mathbf{x})-p_{t-s}(\mathbf{z})$ reduces to $p_{t-s}(\mathbf{z})\times L_{t,\mathbf{x}}(\mathbf{z})$, where

$$L_{t,\mathbf{x}}(s,\mathbf{z}) = -\frac{\mathbf{x}^2}{2(t-s)} + \frac{(\mathbf{z}\cdot\mathbf{x})^2}{2(t-s)^2} + \frac{\mathbf{x}^2}{t-s}\times O\Big(\frac{\mathbf{x}^2}{t-s}+\frac{|\mathbf{z}|^4}{(t-s)^2}\Big). \tag{4.16}$$

Let $D^0_<$, $D^{\rm RH}_<(\mathbf{x}^2)$, $D^{(>)}_{\rm LH}$ and $D^{(<)}_{\rm LH}$ be defined as in Part 3 of the proof of Lemma 3.3 and denote by $I[D^0_<]$, $I[D^{\rm RH}_<(\mathbf{x}^2)]$ etc. be corresponding double integrals on them, so that

$$I[D_<(\mathbf{x}^2)] = I[D^0_<] + I[D^{\rm RH}_<(\mathbf{x}^2)] + I[D^{(<)}_{\rm LH}] + I[D^{(>)}_{\rm LH}]. \tag{4.17}$$

The integrals $I[D^0_<], I[D^{(<)}_{\rm LH}]$ and $I[D^{(>)}_{\rm LH}]$ are independent of $\mathbf{x}$ as the notation suggests.

**2.1.** *Estimation of* $I[D^{\rm RH}_<(\mathbf{x}^2)]$. By definition

$$I[D^{\rm RH}_<(\mathbf{x}^2)] = \int_{t/2}^{t-\mathbf{x}^2} ds \int_{|\mathbf{z}|<\sqrt{4(t-s)\lg(t-s)}} q(\mathbf{z},s)\Big[p_{t-s}(\mathbf{z}-\mathbf{x})-p_{t-s}(\mathbf{z})\Big]d\mathbf{z}. \tag{4.18}$$

Let $(s,\mathbf{z})$ be in the region of integration of this integral. By what is remarked right above, $p_{t-s}(\mathbf{z}-\mathbf{x})-p_{t-s}(\mathbf{z})$ may be replaced by $p_{t-s}(\mathbf{z})L_{t,\mathbf{x}}(s,\mathbf{z})$, which is bounded in absolute value by a constant multiple of

$$p_{t-s}(\mathbf{z})(t-s)^{-1}\mathbf{x}^2\Big[1+(t-s)^{-2}|\mathbf{z}|^4\Big].$$

Keeping this in mind one replaces the upper limit $\sqrt{4(t-s)\lg(t-s)}$ of the inner integral in (4.18) by $\sqrt{8(t-s)\lg\lg(t-s)}$. Since $q(\mathbf{z},s)\le C(\lg|\mathbf{z}|)/s(\lg s)^2$ in view of Corollary 3.1 and the integral of $(\lg|\mathbf{z}|)[1+(t-s)^{-2}\mathbf{z}^4]p_{t-s}(\mathbf{z})$ w.r.t. $\mathbf{z}$ over $|\mathbf{z}|>\sqrt{8(t-s)\lg\lg(t-s)}$ is at most $1/[\lg(t-s)]^2$, the error given rise to by this replacement is bounded above by

$$C'\mathbf{x}^2\int_{t/2}^{t-\mathbf{x}^2}\frac{ds}{s(\lg s)^2[(t-s)(\lg(t-s))^2]} = O\Big(\frac{\mathbf{x}^2}{t(\lg t)^2}\Big).$$

This together with Corollary 3.1 shows that

$$\begin{aligned} I[D^{\rm RH}_<(\mathbf{x}^2)] \;&=\; \int_{t/2}^{t-\mathbf{x}^2} ds \int_{|\mathbf{z}|<\sqrt{8(t-s)\lg\lg(t-s)}} \frac{\lg\mathbf{z}^2}{(\lg s)^2}2\pi p_s(\mathbf{z})\Big[p_{t-s}(\mathbf{z}-\mathbf{x})-p_{t-s}(\mathbf{z})\Big]d\mathbf{z} \\ &\quad + O\Big(\frac{\mathbf{x}^2}{t(\lg t)^2}\Big). \end{aligned} \tag{4.19}$$

Now, substituting the decomposition $\lg\mathbf{z}^2 = \lg(t-s)+\lg[\mathbf{z}^2/(t-s)]$, we have

$$I[D^{\rm RH}_<(\mathbf{x}^2)] = 2\pi\Big[p_t(\mathbf{x})-p_t(0)\Big]\int_{t/2}^{t-\mathbf{x}^2}\frac{\lg(t-s)}{(\lg s)^2}ds + K(\mathbf{x},t) + O\Big(\frac{\mathbf{x}^2}{t(\lg t)^2}\Big), \tag{4.20}$$

where

$$K(\mathbf{x},t) = \int_{t/2}^{t-\mathbf{x}^2}\frac{ds}{(\lg s)^2}\int_{|\mathbf{z}|<\sqrt{8(t-s)\lg\lg(t-s))}}\Big(\lg\frac{\mathbf{z}^2}{t-s}\Big)2\pi p_s(\mathbf{z})p_{t-s}(\mathbf{z})L_{t,\mathbf{x}}(s,\mathbf{z})d\mathbf{z}.$$

On writing $u$ for $t-s$, using the identity

$$2\pi p_u(\mathbf{z})p_{t-u}(\mathbf{z}) = t^{-1}p_{(t-u)u/t}(\mathbf{z}),$$

noting that one may replace $\lg(\mathbf{z}^2/u)$ by $\lg\left(\mathbf{z}^2/[u(t-u)/t]\right)$ in the present estimation, and changing the variables of integration according to $\mathbf{y} = \mathbf{z}/\sqrt{s(t-s)/t}$, the inner integral above becomes

$$\begin{aligned}
&\int_{|\mathbf{z}|<\sqrt{8u\lg\lg u}} \frac{\lg(\mathbf{z}^2/u)}{t} p_{(t-u)u/t}(\mathbf{z}) L_{t,\mathbf{x}}(t-u,\mathbf{z})d\mathbf{z} \\
&= \int_{|\mathbf{y}|<\sqrt{8(t/(t-u))\lg\lg u}} \frac{\lg \mathbf{y}^2}{t}\left[-\frac{\mathbf{x}^2}{2u} + \frac{(t-u)(\mathbf{y}\cdot\mathbf{x})^2}{2ut}\right] p_1(\mathbf{y})d\mathbf{y} + R \\
&= \left(-\frac{1}{2tu}b_1 + \frac{t-u}{4ut^2}b_2\right)\mathbf{x}^2 + O\left(\frac{\mathbf{x}^2}{tu(\lg u)^{3/2}}\right) + R
\end{aligned}$$

(under the restriction $x^2 < u < t/2$), where $R$ denotes the contribution of the error term in $L_{t,\mathbf{x}}(t-u,\mathbf{z})$ and we have used the identity $\int(\mathbf{y}\cdot\mathbf{x})^2 f(|\mathbf{y}|)d\mathbf{y} = 2^{-1}\mathbf{x}^2\int\mathbf{y}^2 f(|\mathbf{y}|)d\mathbf{y}$ valid for a non-negative function $f(r)$ and

$$b_1 = \int_{\mathbf{R}^2}(\lg \mathbf{y}^2)p_1(\mathbf{y})d\mathbf{y} \quad \text{and} \quad b_2 = \int_{\mathbf{R}^2}\mathbf{y}^2(\lg\mathbf{y}^2)p_1(\mathbf{y})d\mathbf{y}.$$

It follows that $b_2 - 2b_1 = 2$ and $|R| \le C|K(\mathbf{x},t)|\mathbf{x}^2/t$. Noting that $\int_{\mathbf{x}^2}^{t/2} u^{-1}(\lg(t-u))^{-2}du = (\lg t/\mathbf{x}^2)(\lg t)^{-2} + O((\lg t)^{-2})$ we infer

$$K(\mathbf{x},t) = \frac{\mathbf{x}^2\lg(t/\mathbf{x}^2)}{2t(\lg t)^2} + O\left(\frac{\mathbf{x}^2}{t(\lg t)^2}\right),$$

and substitution into (4.20) yields

$$I[D_<^{\mathrm{RH}}(\mathbf{x}^2)] = 2\pi\left[p_t(\mathbf{x}) - p_t(0)\right]\int_{t/2}^{t-\mathbf{x}^2}\frac{\lg(t-s)}{(\lg s)^2}ds + \frac{\mathbf{x}^2\lg(t/\mathbf{x}^2)}{2t(\lg t)^2} + O\left(\frac{\mathbf{x}^2}{t(\lg t)^2}\right). \tag{4.21}$$

**2.2.** *Estimation of* $I[D_{\mathrm{LH}}^{(<)}(\mathbf{x}^2)]$. This part is similar to the preceding one but much simpler. This time we substitute $\lg\mathbf{z}^2 = \lg s + \lg(\mathbf{z}^2/s)$ to see that

$$\begin{aligned}
I[D_{\mathrm{LH}}^{(<)}(\mathbf{x}^2)] &= \int_4^{t/2} ds\int_{|\mathbf{z}|<\sqrt{4s\lg\lg s}} q(\mathbf{z},s)\left[p_{t-s}(\mathbf{z}-\mathbf{x}) - p_{t-s}(\mathbf{z})\right]d\mathbf{z} \\
&= 2\pi\left[p_t(\mathbf{x}) - p_t(0)\right]\int_4^{t/2}\frac{ds}{\lg s} + R
\end{aligned} \tag{4.22}$$

with

$$\begin{aligned}
|R| &\le C\int_4^{t/2}\frac{ds}{(\lg s)^2}\int_{\mathbf{R}^2}p_{t-s}(\mathbf{z})\left[\frac{\mathbf{x}^2}{t-s} + \frac{\mathbf{z}^2\mathbf{x}^2}{(t-s)^2}\right]\left(\lg\frac{\mathbf{z}^2}{s}\right)p_s(\mathbf{z})d\mathbf{z} \\
&\le \frac{C'\mathbf{x}^2}{t(\lg t)^2}.
\end{aligned}$$

**2.3.** *Estimation of* $I[D_{\mathrm{LH}}^0(\mathbf{x}^2)]$ and $I[D_{\mathrm{LH}}^{(>)}(\mathbf{x}^2)]$. It is readily checked that $I[D_<^0(\mathbf{x}^2)] = O(1/t(\lg t)^2)$. By the same argument as made for the estimation of $J_{\mathrm{LH}}^{(>)}$ in the proof of Lemma 4.1 we deduce that

$$I[D_{\mathrm{LH}}^{(>)}(\mathbf{x}^2)] = O\left(\frac{\mathbf{x}^2}{t(\lg t)^2}\right). \tag{4.23}$$

**3**. *Completion of Proof.* Combining (4.13), (4.21), (4.22), (4.23) and (4.17) we deduce that

$$\begin{aligned} I[D_<(\mathbf{x}^2)] &= 2\pi\Big[p_t(\mathbf{x}) - p_t(0)\Big]\bigg[\int_{t/2}^{t} \frac{\lg(t-s)}{(\lg s)^2}ds + \int_0^{t/2}\frac{ds}{\lg s}\bigg] \\ &\quad + \frac{\mathbf{x}^2 \lg(t/\mathbf{x}^2)}{2t(\lg t)^2} + O\Big(\frac{\mathbf{x}^2}{t(\lg t)^2}\Big). \end{aligned}$$

Finally, noting $\int_{t/2}^{t} \frac{\lg(t-s)}{(\lg s)^2}ds + \int_0^{t/2}\frac{ds}{\lg s} = t/\lg t + O(t/(\lg t)^2)$ and recalling (4.11) as well as $F_0(t,\mathbf{x}) - F_0(t,0) = I[D_0(\mathbf{x}^2)] + I[D_>(\mathbf{x}^2)] + I[D_<(\mathbf{x}^2)]$ we conclude the formula of Lemma 4.4. $\square$

*Proof of Theorem 1.2.* From Lemmas 4.2 and 4.3 it follows that for $1 < |\mathbf{x}| < M\sqrt{t}$,

$$\frac{F(t,\mathbf{x},1)}{p_t(\mathbf{x})} = \frac{N(\kappa t) - \pi p_t(1) + F_0(t,\mathbf{x}) - F_0(t,0)}{p_t(\mathbf{x})} + O(1).$$

By Lemma 4.4 we can write the right-hand side as

$$2\pi t N(\kappa t) + 2\pi t\Big[-N(\kappa t) + \frac{1}{\lg t}\Big]\frac{p_t(\mathbf{x}) - p_t(0)}{p_t(\mathbf{x})} + \frac{\pi \mathbf{x}^2}{(\lg t)^2}\Big[\lg\Big(\frac{t}{\mathbf{x}^2}\Big) + O(1)\Big] + O(1),$$

in which the second term reduces to $O(\mathbf{x}^2/(\lg t)^2)$ since $N(\kappa t) = 1/\lg t + O(1/(\lg t)^2)$. In view of (4.1) this yields the formula of Theorem 1.2. $\square$

# 5 On the asymptotic expansion related to $N(\lambda)$

Our arguments made in this section are based on the following formula due to C. J. Bouwkamp [2]: for $\lambda > 0, \sigma \geq 0$

$$\int_0^\infty \frac{u^{\sigma-1}e^{-\lambda u}}{(\lg u)^2 + \pi^2}du = \int_0^s \frac{\sin \pi x}{\pi}\Gamma(\sigma + x)e^{-x\lg\lambda}dx + \frac{\theta(\lambda,s)\Gamma(\sigma+s)}{\pi^2\lambda^s}, \tag{5.1}$$

where $s$ may be an arbitrary positive number and $|\theta(\lambda,s)| \leq 1$ (Eq (11) of [2]; see also Dorning et al [4] Eq (17) for the case $\sigma = 0$). From this he obtains the asymptotic expansion of $N(\lambda)$ in powers of $1/\lg\lambda$ as $\lambda \to \infty$. We shall extend the argument of [2]. Take $\sigma = 0$ in (5.1) so that the integral on the left gives $N(\lambda)$ and substitute from Euler's relation $\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x$ as well as $\lambda = \alpha t$ to see that as $t \to \infty$

$$N(\alpha t) = \int_0^s \frac{1}{\Gamma(1-x)\alpha^x}e^{-x\lg t}dx + O\Big(\frac{1}{t^s}\Big). \tag{5.2}$$

**Lemma 5.1** *For any constant $\alpha > 0$, the function $N(\alpha t)$ admits the asymptotic expansion*

$$N(\alpha t) \sim \frac{1}{(\lg t)}\sum_{n=0}^{\infty}\frac{a_n(\alpha)}{(\lg t)^n} \qquad (t \to \infty)$$

*with the constants $a_n(\alpha)$ determined by*

$$\frac{1}{\Gamma(1-x)\alpha^x} = \sum_{n=0}^{\infty}\frac{a_n(\alpha)}{n!}x^n \qquad (|x| < 1). \tag{5.3}$$

*Proof.* The integral on the right-hand side of (5.2) is a Laplace transform, in the variable $\lg t$, of a function regular at $x = 0$ and it is standard to infer that the integral admits an asymptotic expansion in the powers of $1/\lg t$ with the coefficients determined by (5.3), showing the lemma. □

**Lemma 5.2** *For each constant $\alpha > 0$, as $t \to \infty$*

$$\int_0^t N(\alpha s)ds \sim \frac{t}{(\lg t)} \sum_{n=0}^{\infty} \frac{b_n(\alpha)}{(\lg t)^n},$$

$$\int_0^t \Big[-\alpha s N'(\alpha s)\Big]ds \sim \frac{t}{(\lg t)^2} \sum_{n=0}^{\infty} \frac{(n+1)b_n(\alpha)}{(\lg t)^n}$$

*with the constants $b_n(\alpha)$ determined by*

$$\frac{1}{(1-x)\Gamma(1-x)\alpha^x} = \sum_{n=0}^{\infty} \frac{b_n(\alpha)}{n!} x^n \qquad (|x| < 1).$$

*Proof.* Integrate the both sides of (5.2) to see that

$$\begin{aligned}\int_{1/\alpha}^t N(\alpha u)du &= t\int_0^s \frac{1}{(1-x)\Gamma(1-x)\alpha^x} e^{-x \lg t} dx \\ &\quad - \int_0^s \frac{1/\alpha}{(1-x)\Gamma(1-x)} dx + O\Big(\frac{1}{t^{s-1}}\Big),\end{aligned} \tag{5.4}$$

which is valid for every $s > 0$. Note that $1/(1-z)\Gamma(1-z) = 1/\Gamma(-z)$ is an entire function. Since $O(1)$ term is negligible for the asymptotic expansion, we obtain the first formula as in the preceding proof. One can verify the second one in a similar way but by employing (5.1) with $\sigma = 1$. Alternatively, note that the left-hand side integral of the second formula equals $\int_0^t N(\alpha s)ds - tN(\alpha t)$. Then, we readily derive it from the first one combined with the preceding lemma. □

From the Weierstrass product formula

$$\frac{1}{\Gamma(1+x)} = e^{\gamma x} \prod_{n=1}^{\infty} e^{-x/n}\Big(1 + \frac{x}{n}\Big)$$

(cf. [14]) it follows that if $\zeta(z) = \sum_{k=1}^{\infty} k^{-z}$,

$$\lg \frac{1}{\Gamma(1-x)} = -\gamma x - \sum_{n=2}^{\infty} \frac{1}{n}\zeta(n)x^n, \tag{5.5}$$

hence

$$\sum_{n=0}^{\infty} \frac{a_n(\alpha)}{n!} x^n = \exp\Big[-(\gamma + \lg\alpha)x - \sum_{n=2}^{\infty} \frac{1}{n}\zeta(n)x^n\Big].$$

Similarly, using $(1-x)^{-1} = \exp\sum_{n=1}^{\infty} n^{-1}x^n$ we obtain from (5.5)

$$\sum_{n=0}^{\infty} \frac{b_n(\alpha)}{n!} x^n = \exp\Big[-(\gamma + \lg\alpha - 1)x - \sum_{n=2}^{\infty} \frac{1}{n}[\zeta(n) - 1]x^n\Big].$$

From these identities one can readily computes the first several terms of $(a_n(\alpha))_{n=0}^{\infty}$ and those of $(b_n(\alpha))_{n=0}^{\infty}$ as exhibited in (1.2) and (1.3), respectively.

## References


[1] L. Ahlfors, Conformal invariants, Topics in geometric function theory, McGraw Hill, Inc. (1973)

[2] C. J. Bouwkamp, Note on an asymptotic expansion, Indiana Univ. Math. J., **21** (1971), 547-549.

[3] M. van den Berg and E. Bolthausen, On the expected volume of the Wiener sausage for a Brownian bridge, Math. Z. **224** (1997) 33-48.

[4] J. Dorning, B. Nicolaenko and J. Thurber, An integral identity due to Ramanujan which occurs in Neutron transport theory, J. Math. and Mech., **19** (1969) 429-438.

[5] M. D. Donsker and S. R. S. Varadhan, Asymptotics for the Wiener sausage, Comm. Math. Phys., **28** (1975) 525-365

[6] A. Erdélyi, Tables of integral transforms, vol. I, McGraw-Hill, Inc. (1954)

[7] A. Erdélyi, W. Magnus, F. Oberhettinger and F. Tricomi, Higher transcendental functions (Bateman Manuscript Project), vol. III, McGraw-Hill, New York, 1955.

[8] R. K. Getoor, Some asymptotic formulas involving capacity, Z. Whar. Verw. Gebiete, **4** (1965) 248-252.

[9] Y. Hamana, On the expected volume of the Wiener Sausage, J. Math. Soc. Japan, **62** (2010) 1113-1114.

[10] Y. Hamana, and H. Kesten, A large deviation result for the range of random walks and the Wiener sausage, Probab. Theory Rel. Fields, **120** (2001) 183-208.

[11] G. H. Hardy, Ramanujan, Twelve lectures on subject suggested by his life and work, Chelsea, Publishing Company (1940)

[12] K. Itô and H.P. McKean, Jr, Diffusion processes and their sample paths. Springer, (1965)

[13] M. Kac and J. M. Luttinger, Bose-Einstein condensation in the presence of impurities, J. Math. Phys., **14** (1973) 1624-1628.; Bose-Einstein condensation in the presence of impurities II, J. Math. Phys., **15** (1974) 183-186.

[14] N.N. Lebedev, Special functions and their applications, Prentice-Hall, Inc. (1965)

[15] J.F. Le Gall, Sur une conjecture de M. Kac, Probab. Th. Rel. Fields **78** (1988) 389-402.

[16] J.F. Le Gall, Fluctuation results for the Wiener sausage, Ann. Probab., **16** (1988) 991-1018.

[17] J.F. Le Gall, Wiener sausage and self-intersection local time, J. Func. Anal. **88** (1990) 299-341.

[18] I. McGillivray, Large time volume of the pinned Wiener sausage, J. Funct. Anal., **170** (2000) 107-140.

[19] I. McGillivray, The spectral shift function for planar obstacle scattering at low energy, http://arxiv.org/pdf/1111.4377v1.pdf

[20] J. Moser, A Harnack inequality for parabolic differential equations, Comm. Pure Appl. Math. **17** (1964) 101-134; Correction to "A Harnack inequality for parabolic differential equations", Comm. Pure Appl. Math. **20** (1967) 231-236.

[21] F. Spitzer, Some theorems concerning 2-dimensional Brownian motion, TAM. **87** (1958) 187-197.

[22] F. Spitzer, Electrostatic capacity, heat flow and Brownian motion, ZW. **3** (1964) 187-197.

[23] E.C. Titchmarsh, Introduction to the theory of Fourier integrals, 2nd ed., Oxford University Press (1948).

[24] K. Uchiyama, The mean number of sites visited by a random walk, Math. Zeit. **261** (2009), 277-295.

[25] K. Uchiyama, The first hitting time of a single point for random walks. Elect. J. Probab. **16**, no. 71 (2011) 1160-2000.

[26] K. Uchiyama, Asymptotic estimates of the distribution of Brownian hitting time of a disc., J. Theor. Probab., **25** (2012) 450-463. DOI 10.1007/s10959-010-0305-8. / Erratum, J. Theor. Probab. **25**, issue 3 (2012) 910-911, DOI 10.1007/s10959-012-0439-y.

[27] K. Uchiyama, The expected volume of Wiener sausage for the Brownian bridge joining the origin to a parabolic region, (in preparation)

[28] G. E. Uhlenbeck and E. Beth, The quantum theory of the non-ideal gas I. The deviations from the classical theory, Physica **3** (1936) 729-745.

[29] R. Wong, On Laplace tranceforms near the origin. Math. Comp. **29** (1975) 573-576.

[30] W. W. Whitman, Some strong laws for random walks and Brownian motion, PhD thesis, Cornell Univ. (1964)